Ulrich ideals and numerical semigroup rings

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based on the recent works jointly with

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1. Introduction

This talk is based on the recent researches below.

- N. Endo and S. Goto, *Ulrich ideals in numerical semigroup rings of small multiplicity*, arXiv:2111.00498
- N. Endo, S. Goto, S.-i. Iai, and N. Matsuoka, *Ulrich ideals in the ring* $k[[t^5, t^{11}]]$, arXiv:2111.01085

Problem 1.1

Determine all the Ulrich ideals in a given CM local ring.

What is an Ulrich ideal?

- In 1971, J. Lipman investigated **stable maximal ideal** in a CM local ring.
- In 2014, S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, K.-i. Yoshida modified the notion of stable maximal ideal, which they call an **Ulrich ideal**.

Let

- $(A, \mathfrak{m})$ be a CM local ring with $d = \dim A$.
- $\sqrt{I} = \mathfrak{m}$, I contains a parameter ideal Q of A as a reduction (i.e. $I^{n+1} = QI^n$ for some $n \geq 0$)

Definition 1.2 (Goto-Ozeki-Takahashi-Watanabe-Yoshida, 2014)

We say that I is an *Ulrich ideal of A*, if

(1) $I \supsetneq Q$, $I^2 = QI$, and

(2) I/I^2 is A/I-free.

Note that

- (1) $\iff$ $\mathrm{gr}_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is a CM ring with $\mathrm{a}(\mathrm{gr}_I(A)) = 1 - d$.
- If $I = \mathfrak{m}$, then (1) $\iff$ A has minimal multiplicity $\mathrm{e}(A) > 1$.
- (2) and $I \supsetneq Q$ $\implies$ $\mathrm{pd}_A I = \infty$ (Ferrand, Vasconcelos, 1967)

Assume that $I^2 = QI$. Then the exact sequence

$$0 \to Q/QI \to I/I^2 \to I/Q \to 0$$

of A/I-modules shows

$$I/I^2 \text{ is } A/I\text{-free} \iff I/Q \text{ is } A/I\text{-free.}$$

Therefore, if I is an Ulrich ideal of A, then

- $I/Q \cong (A/I)^{\oplus(\mu_A(I)-d)}$,
- $Q :_A I = I$ (i.e., I is a good ideal of A),
- $\mathrm{r}_A(I/Q) = (\mu_A(I) - d) \cdot \mathrm{r}(A/I) = \mathrm{r}(A)$

so that

$$d + 1 \leq \mu_A(I) \leq d + \mathrm{r}(A).$$

Hence, when A is a Gorenstein ring,

every Ulrich ideal I is generated by $d + 1$ elements (if it exists).

For every Ulrich ideal I of A, we have

Theorem 1.3 (Goto-Takahashi-T, 2015)

$\mathrm{Ext}^i_A(A/I, A)$ *is* A/I*-free for* $\forall i \in \mathbb{Z}$.

Hence

$$\mu_A(I) = d + 1 \iff \text{G-dim}_A A/I < \infty.$$

This shows if A is G-regular, then $\mu_A(I) \geq d + 2$.

Consequently, if I is an Ulrich ideal of A with $\mu_A(I) = d + 1$, then

- A/I is Gorenstein $\iff$ A is Gorenstein,
- I is a totally reflexive A-module,
- $\mathrm{pd}_A I = \infty$, and

the minimal free resolution of I has a very restricted form.

In what follows, assume $d = 1$ and I is an Ulrich ideal of A with $\mu_A(I) = 2$.

Write $I = (a, b)$, where $a, b \in A$ and $Q = (a)$ is a reduction of I.

By taking $c \in I$ with $b^2 = ac$, the minimal free resolution of I has the form

$$\cdots \longrightarrow A^{\oplus 2} \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} A^{\oplus 2} \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} A^{\oplus 2} \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} I \longrightarrow 0$$

We then have $I = J$, once

$$\mathrm{Syz}_A^i(I) \cong \mathrm{Syz}_A^i(J) \quad \text{for some } i \geq 0$$

provided I, J are Ulrich ideals of A. (GOTWY, 2014)

Corollary 1.4 (GOTWY, 2014)

Suppose that A is a Gorenstein ring. If I, J are Ulrich ideals of A with $\mathfrak{m}J \subseteq I \subsetneq J$, then A is a hypersurface.

Let $\mathcal{X}_A$ be the set of Ulrich ideals in A.

On the other hand

- If A has finite CM representation type, then $\mathcal{X}_A$ is finite. (GOTWY, 2014)
- Suppose that $\exists$ a fractional canonical ideal K. Set $\mathfrak{c} = A : A[K]$.

  If A is a non-Gorenstein almost Gorenstein ring, then

  $$\mathcal{X}_A \subseteq \{\mathfrak{m}\} \qquad \text{(GTT, 2015)}$$

  If A is a 2-almost Gorenstein ring with minimal multiplicity, then

  $$\{\mathfrak{m}\} \subseteq \mathcal{X}_A \subseteq \{\mathfrak{m}, \mathfrak{c}\} \qquad \text{(Goto-Isobe-T, 2020)}$$

We expect that there is a strong connection between

the behavior of Ulrich ideals and the structure of base rings.

Problem 1.1

Determine all the Ulrich ideals in a given CM local ring.

Question 1.5

How many two-generated Ulrich ideals are contained in a given numerical semigroup ring?

Let

- $0 < a_1, a_2, \dots, a_\ell \in \mathbb{Z}$ s.t. $\gcd(a_1, a_2, \dots, a_\ell) = 1$
- $H = \langle a_1, a_2, \dots, a_\ell \rangle = \left\{ \sum_{i=1}^{\ell} c_i a_i \;\middle|\; 0 \le c_i \in \mathbb{Z} \text{ for all } 1 \le i \le \ell \right\}$
- $A = k[[H]] = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]] \subseteq V = k[[t]] = \overline{A}$, where k is a field
- $\mathrm{c}(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for all } m \in \mathbb{Z} \text{ s.t. } m \ge n\}$

Note that $t^{\mathrm{c}(H)} V \subseteq A$.

2. Method of computation

Previous Method

Let

- $(A, \mathfrak{m})$ be a Gorenstein local ring with $\dim A = 1$,
- $\mathcal{X}_A$ be the set of Ulrich ideals in A,
- $\mathcal{Y}_A$ be the set of birational module-finite extensions B of A

  (i.e., $A \subseteq B \subseteq \mathrm{Q}(A)$ and B is a finitely generated A-module)

  s.t. B is a Gorenstein ring and $\mu_A(B) = 2$.

Then, there exists a bijective correspondence

$$\mathcal{X}_A \to \mathcal{Y}_A, \quad I \mapsto A^I$$

where

$$A^I = \bigcup_{n \geq 0} [I^n : I^n] = I : I.$$

Let

- $V = k[[t]]$ be the formal power series ring over a field k
- A be a k-subalgebra of V.

We say that

$$A \text{ is a } \textit{core of } V \quad \overset{def}{\Longleftrightarrow} \quad t^c V \subseteq A \text{ for some } c \gg 0.$$

Example 2.1

- $k[[H]]$ is a core of V,
- $A = k[t^2 + t^3] + t^4 V$ is core, but $A \neq k[[H]]$ for any numerical semigroup H.

Let A be a core of V and suppose $t^c V \subseteq A$ with $c \gg 0$. Then

$$k[[t^c, t^{c+1}, \dots, t^{2c-1}]] \subseteq A \subseteq V$$

so that V is a birational module-finite extension of A.

Hence, for every core A of V,

- $V = \overline{A}$
- A is a CM complete local domain with $\dim A = 1$
- $V/\mathfrak{n} \cong A/\mathfrak{m}$

where $\mathfrak{m}$ (resp. $\mathfrak{n} = tV$) stands for the maximal ideal of A (resp. V).

Let $\mathrm{o}(-)$ denote the $\mathfrak{n}$-adic valuation of V, and set

$$H = v(A) = \{\mathrm{o}(f) \mid 0 \neq f \in A\}.$$

Note that

$H = v(A)$ is symmetric $\iff$ A is Gorenstein (Kunz, 1970)

Let I be an Ulrich ideal of A with $\mu_A(I) = 2$. Choose $f, g \in I$ s.t. $I = (f, g)$ and $I^2 = fI$. Then

$$A^I = I : I = \frac{I}{f} = A + A \cdot \frac{g}{f}$$

is a core of V, and $v(A^I)$ is symmetric if A is Gorenstein.

Lemma 2.2 (Key Lemma)

Let I be an Ulrich ideal in A with $\mu_A(I) = 2$. Then one can choose $f, g \in I$ satisfying the following conditions, where $a = \mathrm{o}(f)$ and $b = \mathrm{o}(g)$.

(1) $I = (f, g)$ and $I^2 = fI$.

(2) $a, b \in H$ and $0 < a < b < a + \mathrm{c}(H)$.

(3) $b - a \notin H$, $2b - a \in H$, and $a = 2 \cdot \ell_A(A/I)$.

(4) If $a \geq \mathrm{c}(H)$, then $\mathrm{e}(A) = 2$ and $I = A : V$.

■ Method of computation

- Step 1 ⋯ Let $I \in \mathcal{X}_A$ with $\mu_A(I) = 2$. Choose $f, g \in I$ which satisfy the conditions in Lemma 2.2.
- Step 2 ⋯ Consider $A^I = A + A \cdot \frac{g}{f}$ and determine $v(A^I)$.
- Step 3 ⋯ Determine the possible pair $(\mathrm{o}(f), \mathrm{o}(g))$.
- Step 4 ⋯ Determine the form of generators of I.
- Step 5 ⋯ Conversely, the ideal of the form as in Step 4 is an Ulrich ideal.

3. Main theorem

Theorem 3.1 (Main theorem)

Let $\ell \geq 7$ *be an integer such that* $\gcd(3, \ell) = 1$ *and set* $A = k[[t^3, t^\ell]]$*.*

(1) *Suppose that* $\ell = 3n + 1$ *where* $n \geq 3$ *is odd. Let* $q = \frac{n-1}{2}$*. Then*

$$\begin{aligned}\mathcal{X}_A &= \left\{ \left(t^\ell + \sum_{j=1}^{q} \alpha_j t^{\ell+3j-1}, t^{\ell+3q+2} \right) \,\middle|\, \alpha_1, \alpha_2, \dots, \alpha_q \in k \right\} \\ &\bigcup \left\{ \left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \,\middle|\, 1 \leq i \leq q, \alpha_0, \dots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}.\end{aligned}$$

(2) *Suppose that* $\ell = 3n + 1$ *where* $n \geq 2$ *is even. Let* $q = \frac{n}{2}$*. Then*

$$\mathcal{X}_A = \left\{ \left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \,\middle|\, 1 \leq i \leq q, \alpha_0, \dots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}.$$

Theorem 3.1 (continued)

(3) Suppose that $\ell = 3n+2$ where $n \geq 1$ is odd. Let $q = \frac{n-1}{2}$. Then

$$\mathcal{X}_A = \left\{ \left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \,\middle|\, 1 \leq i \leq q, \alpha_0, \dots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}.$$

(4) Suppose that $\ell = 3n+2$ where $n \geq 2$ is even. Let $q = \frac{n}{2}$. Then

$$\begin{aligned} \mathcal{X}_A = & \left\{ \left(t^{\ell} + \sum_{j=1}^{q} \alpha_j t^{\ell+3j-2}, t^{\ell+3q+1} \right) \,\middle|\, \alpha_1, \alpha_2, \dots, \alpha_q \in k \right\} \\ & \bigcup \left\{ \left(t^{6i} + \sum_{s=0}^{i-1} \alpha_s t^{\ell+3s}, t^{\ell+3i} \right) \,\middle|\, 1 \leq i \leq q, \alpha_0, \dots, \alpha_{i-1} \in k, \alpha_0 \neq 0 \right\}. \end{aligned}$$

Moreover, the coefficients α_i's in the system of generators of $I \in \mathcal{X}_A$ are uniquely determined for I.

We denote by $\mathcal{X}_A^g$ the set of Ulrich ideals in A generated by monomials in t.

Then $\mathcal{X}_A^g$ is a finite set (GOTWY, 2014).

Corollary 3.2

Let $\ell \geq 7$ be an integer s.t. $\gcd(3, \ell) = 1$ and set $A = k[[t^3, t^\ell]]$. Then

(1) $\mathcal{X}_A \neq \emptyset$.

(2) $\mathcal{X}_A$ *is finite* $\iff$ k *is a finite field.*

(3) $\mathcal{X}_A^g = \emptyset$ $\iff$ $\ell = 3n + 1$ *or* $\ell = 3n + 2$ *for some even integer* $n \geq 2$

Example 3.3

Let $A = k[[t^3, t^7]]$. Then

$$\mathcal{X}_A = \{(t^6 + \alpha t^7, t^{10}) \mid 0 \neq \alpha \in k\}.$$

Hence, ${}^\#\mathcal{X}_A = {}^\# k - 1$ and A does not contain monomial Ulrich ideals.

Thank you for your attention.